Integrals of Power Series

The integral

$$\int e^{-x^2}\,dx$$

has no simple anti-derivative. We have considered numerical integration in Sections 7.5 and 7.6 of the textbook, but this is not the only way to approximate this integral. Power series provide a valuable alternative.

The steps below outline how to develop an evaluation method for integrals using power series. Your report should contain these parts, plus any additional material you need to produce a coherent explanation of the method.

1. Consider a power series

$$P(x) = C_0 + C_1 x + \cdots + C_n x^n + \cdots$$

   By taking the antiderivative of each term, find a new power series. You may assume that the antiderivative of $P(x)$ is this new power series.

2. Suppose that $P(x)$ has radius of convergence R. Use the Ratio Test to show that your new power series has the same radius of convergence.

3. Pick one or two examples of power series with finite (and positive) radius of convergence. Show, both graphically and algebraically, that your formula for the anti-derivative power series is correct and that it has the same radius of convergence.

The second part of the project is to work out how quickly the antiderivative power series converges to the right answer, and to compare this to how quickly Simpson's rule converges to the right answer.

As a warmup, consider an alternating series

$$S = a_1 - a_2 + a_3 - \cdots + (-1)^{n-1} a_n + \cdots$$

satisfying $a_{n+1} \le a_n$ for all n and $\lim_{n\to\infty} a_n = 0$. Let S_n be the nth partial sum, i.e.,

$$S_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n-1} a_n$$

Your group should show that

$$|S_n - S| \le a_{n+1}.$$

Argue this both analytically and graphically.

Next, we find a similar estimate for the power series for $F(x) = \int_0^x e^{-t^2} dt$ centered at $x = 0$. First find this power series. You may want to start by taking the power series expansion for e^x and use this to obtain the power series of e^{-x^2}, explaining your steps.

Let $P_n(x)$ be the $(n+1)$-st partial sum of power series of $F(x)$ at $x = 0$. Let $E_n(x) = |P_n(x) - F(x)|$ be the error between the approximation $P_n(x)$ and the exact function $F(x)$. Derive an estimate for $E_n(x)$ for $x \geq 0$.

Use this estimate to determine an estimate on the error $E_n(x)$ for $x \leq 0$. You must carefully explain your argument, showing your reasoning. For $n = 5, 10, 20, 40, 80, 160$, estimate the error of approximation $E_n(1)$ at $x = 1$, based on your work so far. Give an approximation for $\int_0^1 e^{-x^2} dx$ that has at least 20 digits accuracy.

Then compare your estimates to SIMP(n) for the same values of n with the integral $F(1) = \int_0^1 e^{-x^2}\, dx$. Write a paragraph to compare the two methods in terms of the number of numerical operations needed in relation to their approximation accuracy, and any other aspects you deem worth comparing.

Remember that **presentation counts!** Your report should have a title page with your recitation section information, date, all members names, both typed and signed. While it must have introductory and concluding paragraphs, be concise and to the point. If at all possible, type the report the best possible way you can. Don't repeat yourself; use correct grammar and complete sentences. Proof read your final report several times. Organize all relevant numerical data in tables; do not spread them throughout the text.

A few more suggestions: use sections, each dealing with a specific topic. However, do not use leading sentences such as "1. We are asked to do.... We found the answer to be ..."; or "Problem #..., Solution: ..." In particular, do not repeat in your report the numbering we have used above. Do your best to make your report flow smoothly. Present your ideas clearly and concisely, make your point and then move on. Repetition, meaningless verbiage, and jumbles of calculations are all bad. Use an appendix for long or difficult calculations, or for examples of programs used. Imagine that your reader took Math 107 several years ago but does not remember all the details; be prepared to explain tricky points. If your report reads like a section of your textbook, you probably have it right. If it reads like a badly done homework question, then you need to fix it. Poor presentation will hurt your grade!